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Absence of Self-Averaging of the Order Parameter in the Sherrington–Kirkpatrick Model

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We prove that if $\hat{H}_N$ is the Sherrington–Kirkpatrick (SK) Hamiltonian and the quantity $\bar{q}_N = N^{-1} \sum \langle S_i \rangle_H^2$ converges in the variance to a nonrandom limit as $N \to \infty$, then the mean free energy of the model converges to the expression obtained by SK. Since this expression is known not to be correct in the low-temperature region, our result implies the "non-self-averaging" of the order parameter of the SK model. This fact is an important ingredient of the Parisi theory, which is widely believed to be exact. We also prove that the variance of the free energy of the SK model converges to zero as $N \to \infty$, i.e., the free energy has the self-averaging property.

KEY WORDS: Spin glasses; order parameter; self-averaging.

1. INTRODUCTION

The Sherrington–Kirkpatrick (SK) model[(1)] is one of the widely accepted models of disordered spin systems with a random competitive interaction. This model is defined by the Hamiltonian

$$\hat{H}_N = -N^{-1/2} \sum_{1 \leqslant i < j \leqslant N} J_{ij} S_i S_j \tag{1.1}$$

where the spins $S_1, \ldots, S_N$ take values ± 1 (Ising spins) and the J_{ij}, $1 \leqslant i < j \leqslant N$, are independent identically distributed Gaussian random variables with zero mean and variance J^2.[2] It is believed that in the thermodynamic limit $N \to \infty$ this model gives answers coinciding with those

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which would appear in the limit of the infinite-interaction range or infinite dimensionality d of more realistic finite-range models, defined on the lattice $\mathbb{Z}^d$.[3]

By using the so-called replica trick, Sherrington and Kirkpatrick[1] found the following expression for the mean free energy in the thermodynamic limit:

$$\beta f_{\mathrm{SK}} = -K^2/4(1-q)^2 - (2\pi)^{-1/2} \int \ln 2\,\mathrm{ch}(Kq^{1/2}z)\, e^{-z^2/2}\, dz \tag{1.2}$$

$$q = (2\pi)^{-1/2} \int \mathrm{th}^2\,(Kq^{1/2}z)\, e^{-z^2/2}\, dz \tag{1.3}$$

where $K^2 = \beta^2 J^2$ and β is the inverse temperature. This expression predicts the phase transition into the spin-glass state at the temperature $T_f = J$. However, this "SK solution" cannot be correct in the most interesting low-temperature region $T < T_f$, since it does not satisfy general and important requirements such as the nonnegativity of the entropy and magnetic susceptibility, some stability conditions, etc.[3,4]

The SK model has been considered in numerous physics papers (see refs. 3 and 4 and references therein), in which the rich and complex structure of this model was discovered and studied. The physical theory developed contains a number of new fundamental concepts and facts which have no analogs in nonrandom systems and can be applied to a wide range of complex systems.

One of the interesting new objects of the spin-glass theory is the Edwards–Anderson order parameter q, which, according to one interpretation, is the thermodynamic limit of the second moment q_N of the random field of the magnetization $m_i = \langle S_i \rangle_H$. (The symbols $\langle \cdots \rangle_H$ and $E\{\cdots\}$ will denote respectively, the average over $S_1,..., S_N$ with the Gibbs measure of (1.1) and the average over J_{ij}). The conventional wisdom of statistical mechanics and the theory of disordered systems suggests that the random (due to a randomness of J_{ij}) variable

$$\hat{q}_N = N^{-1} \sum_{i=1}^{N} \langle S_i \rangle_{H_N}^2 \tag{1.4}$$

should tend to a nonrandom limit and this limit should be equal to q. One possible formalization of this wisdom is the following: if

$$q_N = E\{\hat{q}_N\} = E\{\langle S_i \rangle_{H_N}^2\} \tag{1.5}$$

[3] For the former limit this was proved if the temperature is large enough. $\beta J \ll 1$.[2]

and

$$\Delta_N = E\{(\hat{q}_N - q_N)^2\} \tag{1.6}$$

then

$$\lim_{N\to\infty} q_N = q \tag{1.7}$$

and

$$\lim_{N\to\infty} \Delta_N = 0 \tag{1.8}$$

We will call such a property the self-averaging. This property, for instance, is satisfied by the free energy of non-mean-field-like models [i.e., models that, unlike (1.1), do not contain N explicitly in their Hamiltonians] under fairly weak conditions on the decay of the interaction[5-7] and a variety of quantities in disordered one-body theory.[8]

However, according to the Parisi theory,[4] which is widely believed to be exact, the quantity $\hat{q}_N$ does not have the self-averaging property. In particular,

$$\lim_{N\to\infty} \Delta_N = 1/3 \left\{ \int_0^1 q^2(x)\,dx - \left[\int_0^1 q(x)\,dx \right]^2 \right\}$$

where $0 \leqslant q(x) \leqslant 1$ is a functional order parameter of the Parisi theory. Since $q(x)$ is not a constant for $T < T^f = J$, the rhs of (1.8) is strictly positive in this whole low-temperature region. This remarkable fact is connected in the Parisi theory with the exponentionally large number and rich structure of the pure equilibrium states or "valleys" for the Hamiltonian (1.1) for $N \to \infty$.

The Parisi theory of the spin-glass phase transition is fairly convincing and interesting from the theoretical physics point of view, but has not acquired the status of rigorous mathematical physics. The rigorous papers devoted to the study of the SK model[2,10] contain mainly the treatment of the high-temperature region $T > T_f$. Besides, some explicit bounds pertaining to the low-temperature region were obtained which imply the existence of a phase transition at $T = J$.[10] In the present paper we prove rigorously a statement that might be considered as the statement of the absence of the self-averaging property of quantity (1.4). Namely, we show, that assuming (1.8), we come with necessity to the Sherrington–Kirkpatrick expression (1.2)–(1.3) for the free energy of the model. Since, according to above discussion (see also refs. 3 and 4), this expression cannot be correct for $T < J$, we should conclude that q_N is not a self-averaging quantity. This result might be considered as rigorous support of the Parisi theory.

Our method consists in performing the thermodynamic transition $N \to \infty$ step by step. We study and control changes in the respective quantities after adding the Nth spin S_1 to the system of $N-1$ spins $S_2, ..., S_{N-1}$. Similar arguments are already known in this field as the cavity method,[4] and our approach can be considered as a rigorous version of these arguments. An analogous method was used by Pastur[9] to find the integrated density of states of some ensembles of random matrices, including, in particular, the ensemble of the Gaussian matrices $N^{-1/2} J_{ij}$ from (1.1).

The paper is organized as follows. In Section 2 we formulate our main result (Theorem 1) and give its proof modulo several important technical statements. These statements are proved in Section 3. The Appendix contains the proof of disappearance of the variance of the free energy f_N of the Hamiltonian (1.1) in the thermodynamic limit $N \to \infty$, i.e., the analog of (1.8) for f_N. Our initial intention was just to mention the validity of this property for the free energy, in contrast to its absence for q_N. To our surprise we did not find in the literature a rigorous proof of this property. Moreover, the proof of the self-averaging of the free energy of the SK model was formulated as a problem in ref. 10. Our proof, which is valid for all $T > 0$, is based on a form of the martingale ideology, also borrowed from the spectral theory of random matrices.[9,11]

2. THE MAIN RESULTS

Denote by $G(v_1, ..., v_N) = G(v)$ the solution of the equation

$$-1/2 \sum_{i=1}^{N} \frac{d^2 G_N}{dv_i^2} + G_N(v) = \delta(v), \qquad v \in \mathbb{R}^n \tag{2.1}$$

It is easy to show that

$$G(v) = \int dt (2\pi t)^{-N/2} e^{-t - v^2/2t} \tag{2.2}$$

Thus, $G(v)$ is positive, symmetric, and

$$\int_{\mathbb{R}^N} dv\, G(v) = 1, \qquad \int_{\mathbb{R}^N} dv\, G(v)\, v_i^2 = 1 \tag{2.3}$$

Consider now the more general Hamiltonian

$$H_N = \hat{H}_N - N^{-1/4} \sum_{i=1}^{N} v_i S_i \tag{2.4}$$

where H_N is given by (1.1) and $v = (v_1,..., v_N)$ are random variables independent of the J_{ij} and having a probability distribution with the density (2.2). Thus, the v_i enter (2.4) as a random external field which is infinitesimal for $N \to \infty$. This term plays an important role in our analysis as a symmetry-breaking field. In particular, this term allows us to define a nonzero random field of local magnetization

$$m_i = \langle S_i \rangle_{H_N}, \qquad i = 1,..., N \tag{2.5}$$

Let $\hat{f}_N$ and f_N be the free energies of the Hamiltonians (1.1) and (2.4):

$$\hat{f}_N = -(\beta N)^{-1} \ln \hat{Z}_N, \qquad f_N = -(\beta N)^{-1} \ln Z_N \tag{2.6}$$

where $\hat{Z}_N$ and Z_N are the respective partition functions. By using (2.3) and (2.4), it is easy to show that

$$\left| \hat{f}_N - \int_{\mathbb{R}^N} dv\, G(v)\, f_N \right| \leqslant \beta N^{-1/4} \tag{2.7}$$

In particular, combining this bound and Theorem 2, proved in the Appendix, we obtain that

$$\lim_{N \to \infty} E\left\{ \left| \hat{f}_N - \int_{\mathbb{R}^N} dv\, G(v)\, E\{f_N\} \right| \right\} = 0 \tag{2.8}$$

Thus, adding the v term to the Hamiltonian H_N, we do not change the free energy in the thermodynamic limit. This fact seems to be a natural necessary condition for the external field to be called a symmetry-breaking one.

However, we should admit that our symmetry-breaking field is somewhat special and unusual, not only because of the special form of the distribution (2.2) that will be fairly important below (see the proofs of Lemmas 3.2 and 3.3), but also because its strength $h_N = N^{-1/4}$ depends explicitly on N. The standard way to handle a symmetry-breaking field in statistical mechanics is to keep its strength h fixed during the thermodynamic limit transition and send h to zero only after performing this limit.

Denote by E_G the expectation over J_{ij} and v_i, i.e.,

$$E_G\{\cdots\} = \int E\{\cdots\}\, G_N(v)\, dv \tag{2.9}$$

Theorem 1. Let

$$\bar{q}_N = N^{-1} \sum_{i=1}^{N} \langle S_i \rangle_{H_N}^2$$

$$q_N(\beta) = E_G\{q_N\}, \qquad \Delta_N(\beta) = E_G\{[q_N - q_N(\beta)]^2\} \tag{2.10}$$

Suppose that for fixed J and β, $\beta J > 1$, there exists $\varepsilon > 0$ such that uniformly in $\beta \in (\beta - \varepsilon, \beta]$

$$\underline{\lim}_{N \to \infty} q_N(\beta) > 0 \tag{2.11}$$

$$\lim_{N \to \infty} \Delta_N(\beta) = 0 \tag{2.12}$$

Then, for these β and J the thermodynamic limit of the mean free energy exists:

$$\lim_{N \to \infty} E_G\{f_N\} = f \tag{2.13}$$

and coincides with the Sherrington–Kirkpatrick free energy f_{SK} given by (1.2) and (1.3).

Proof. Let us introduce the Hamiltonian

$$H_N(t) = -N^{-1/2} \sum_{i=2}^{N} \sum_{i<j} J_{ij} S_i S_j - \sum_{i=2}^{N} (N^{-1/2} J_{1i} S_1 t + N^{-1/4} v_i) S_i - N^{-1/4} v_1 S_1 \tag{2.14}$$

where $0 \leqslant t \leqslant 1$. In $H_N(t)$ the interaction of the spin S_1 with all other spins $S_2,..., S_N$ has the varying strength $t \in [0, 1]$. Thus,

$$H_N(1) = H_N, \qquad H_N(0) = \alpha_N^2 H_{N-1} + \gamma_N \sum_{2}^{N} v_i S_i - N^{-1/4} v_1 S_1 \tag{2.15}$$

where

$$\alpha_N = (1 - N^{-1})^{1/4}; \qquad \gamma_N = N^{-1/4}(\alpha_N - 1) = O(N^{-5/4}) \tag{2.16}$$

We shall see later that for all quantities of interest, the second and third terms in $H_N(0)$ are negligibly small. Hence, setting $t = 0$, we obtain the system of $N-1$ spins $S_2,..., S_N$ at the temperature

$$\beta_N = \alpha_N^2 \beta \to \beta, \qquad N \to \infty \tag{2.17}$$

Set

$$x_n^{(N)}(t) = E_G\{m_1^{2n}(t)\} \tag{2.18a}$$

$$m_1(t) = \langle S_1 \rangle_{H_N(t)} \tag{2.18b}$$

By differentiation of $x_n^N(t)$ with respect to t and using the simple identity

$$E\{J_{ij}, \varphi(J_{ij})\} = J^2 E\{\varphi'(J_{ij})\} \tag{2.19}$$

which is valid for the Gaussian random variables J_{ij}, $E\{J_{ij}\}=0$, $E\{J_{ij}^2\}=J^2$, we obtain

$$x_n^{(N)}(t) = -K^2 t q_{N-1}(\beta_N)(\hat{\alpha}x^{(N)})_n + t a_n^{(N)} + t r_n^{(N)} \tag{2.20}$$

where

$$(\hat{\alpha}x)_1 = 2(-x_2+2x_1)+4(x_1-x_2) \tag{2.21}$$

$$(\hat{\alpha}x)_n = 2n(2n-1)(-x_{n-1}-x_{n+1}+2x_n)+4n(x_n-x_{n+1}), \qquad n>1$$

$$a_1^{(N)} = -2q_{N-1}(\beta_N)\,K^2, \qquad a_n^{(N)}=0, \qquad n>1 \tag{2.22a}$$

$$\begin{aligned} r_n^{(N)} = K^2 E_G\{&2n(2n-1\}\, m_1^{2n-2}[\delta_N(t)+2m_1\varepsilon_N(t) \\ &+m_1^2\kappa_N(t)] + 4nm_1^{2n-1}[\varepsilon_N(t)+m_1^2\kappa_N(t)] \\ &+N^{-1/4}v_1(1-m_1^2) \end{aligned} \tag{2.22b}$$

and

$$\delta_N(t) = N^{-1}\sum_{i=2}^{N} m_i^2 - q_{N-1}(\beta) \tag{2.23a}$$

$$\varepsilon_N(t) = N^{-1}\sum_{i=2}^{N} m_i\langle S_i S_1\rangle_{H_N(t)} - m_1 q_{N-1}(\beta_N) \tag{2.23b}$$

$$\kappa_N(t) = N^{-1}\sum_{i=2}^{N} m_i\langle S_1 S_i\rangle_{H_N(t)}^2 - m_1^2 q_{N-1}(\beta_N) \tag{2.23c}$$

where $m_i = m_i(t)$ is defined by (2.5) with $H_N(t)$ instead of H_N. By using the definition (2.23), the Schwartz inequality, and the Proposition of Section 3, we obtain the estimate

$$E_G\{|r_n^{(N)}|\} \leqslant K^2 n^2 [x_{4n-4}^{(N)}]^{1/2}\,\lambda_N \tag{2.24}$$

where λ_N here and below denotes a quantity which is independent of n, K, and t, and tends to zero as $N\to\infty$.

Consider now the operator $\hat{A}$, defined by the operation (2.21) in the Hilbert space $\mathscr{H}$ of semi-infinite sequences $x=\{x_n\}_{n\geqslant 1}$ with the norm

$$\|x\|^2 = \sum_{n=1}^{\infty} n^{-1}\,|x_n|^2 \tag{2.25}$$

According to Lemma 3.4 of Section 3, $\hat{A}$ is a nonnegative and selfadjoint operator in $\mathscr{H}$ and according to Lemma 3.5, $r^{(N)} = \{r_n^{(N)}\}_{n \geqslant 1}$ belongs to $\mathscr{H}$ uniformly in $N \to \infty$. Thus, we can rewrite (2.20)–(2.22) as the Cauchy problem in $\mathscr{H}$:

$$\frac{dx^{(N)}}{dt} = -Kq_{N-1}(\beta_N)\, tAx^{(N)} + ta^{(N)} + tr^{(N)} \tag{2.26}$$

$$x^{(N)}(0) = E_G\{\text{th}^{2n}(\beta N^{-1/4} v_1)\} \tag{2.27}$$

It is easy to check that

$$\|x^{(N)}(0)\|^2 \leqslant E_G\{\ln \text{ch}^2(\beta N^{-1} v_1)\} \tag{2.28}$$

and that

$$y_n^{(N)}(t) = \int_{-\infty}^{\infty} \text{th}^{2n}[Kq_{N-1}^{1/2}(\beta_N)\, tu + \beta N^{-1/4} v_1]\, e^{-u^2/2}(2\pi)^{-1/2}\, du \tag{2.29}$$

is the solution of the Cauchy problem (2.26) and (2.27) with $r^{(N)} = 0$. Moreover, according to (2.24) and Lemma 3.5, $\|r^{(N)}\| \to 0$ as $N \to \infty$. Therefore,

$$\lim_{N \to \infty} \|x^{(N)} - y^{(N)}\| = 0 \tag{2.30}$$

This relation implies in particular that

$$q_N(\beta) = x_1^{(N)}(1) = \\ = \text{th}^{2n}[Kq_{N-1}^{1/2}(\beta_N)\, u]\, e^{-u^2/2}(2\pi)^{-1/2}\, du + \lambda_N \tag{2.31}$$

where $\lambda_N \to 0$ as $N \to \infty$. On the other hand, according to (2.10), (2.15), and the Proposition of Section 3,

$$q_n(\beta) = E_G\left\{N^{-1} \sum_{i=1}^{N} \langle S_i \rangle^2_{H_N(1)}\right\} = \\ = q_{N-1}(\beta) + \delta_N(1) + O(N^{-1}) = q_{N-1}(\beta_N) + o(1), \qquad N \to \infty \tag{2.32}$$

Hence, we can write (2.31) in the form

$$q_N(\beta) = \int \text{th}^2[Kq_N^{1/2}(\beta)\, u]\, e^{-u^2/2}(2\pi)^{-1/2}\, du$$

which implies that every limit point q of the sequence $\{q_N(\beta)\}$ satisfies Eq. (1.3). According to our assumption, any such point is strictly positive. Since Eq. (1.3) has a unique nonzero solution, we see that the sequence $\{q_N(\beta)\}$ has a limit which coincides with this solution. Thus, in the thermodynamic limit the order parameter $q_N(\beta)$ of (2.10) coincides with the order parameter considered by Sherrington and Kirkpatrick.[1] Let us show now that under our assumptions (2.11) and (2.12) the free energy is also equal to the SK expression (1.2)–(1.3).

Denote by $Z_N(t)$ the partition function of the Hamiltonian (2.14). Then, by using (2.19) and taking into account the Proposition from Section 3, we obtain for

$$F_N(t) = E_G\{\ln Z_N(t)\} \tag{2.33}$$

$$\begin{aligned} \frac{dF_N}{dt} &= \beta N^{-1/2} E_G\left\{\sum_{i=2}^{N} J_{ik}\langle S_1 S_k\rangle_{H_N(t)}\right\} \\ &= K^2 t E_G\left\{N^{-1}\sum_{i=2}^{N}(1-\langle S_1 S_k\rangle_{H_N(t)}\right\} \\ &= K^2 t[1-y_1(t)\, q] + o(1) \end{aligned} \tag{2.34}$$

where, according to (2.31),

$$y_1(t) = \int \operatorname{th}^2(Kq^{1/2}ut)\, e^{-u^2/2}(2\pi)^{-1/2}\, du \tag{2.35}$$

In addition, according to (2.15) and the Proposition of Section 3,

$$\begin{aligned} F_N(0) &= F_{N-1}(1) - K^2/4E_G\left\{(N-1)^{-1}\sum_{k=3}^{N}(1-\langle S_2 S_k\rangle_{H_{N-1}(1)})\right\} \\ &= F_{N-1}(1) - K^2/4(1-q^2) + o(1) \end{aligned} \tag{2.36}$$

Combining (2.34) and (2.36), we obtain the relation

$$F_N(1) = F_{N-1}(1) - K^2\int_0^1 t[1-y_1(t)\, q]\, dt - K^2/4(1-q^2) + o(1)$$

which in view of (2.8) and (2.33) shows that the mean free energy

$$E_G\{f\} = -(\beta N)^{-1} E_G\{\ln Z_N\} = -(\beta N)^{-1} F_N(1)$$

tends to the limit

$$f = K^2/4t(1-q^2) - 1/\beta\int_0^1 [1-y_1(t)\, q]\, dt \tag{2.37}$$

Consider now the function

$$\phi(t) = -K^2(1-q^2)/4 + \int_0^t [1 - y_1(\tau)]\, d\tau \tag{2.38}$$

By differentiating (2.38) and using (2.35), we find that

$$\phi(t) = -K^2(1-q^2)/4 + K^2(1-q)\, t^2/2$$
$$+ \int \ln 2\, \mathrm{ch}(Kq^{1/2}ut)\, e^{-u^2/2}(2\pi)^{-1/2}\, du$$

But according to (2.37) and (2.38), $f = -\beta^{-1}\phi(1)$, and as a result, we obtain formula (1.2) for f.

Remarks. 1. We considered the SK Hamiltonian (1.1) without an external field [the infinitesimal field in (2.4) played the role of a symmetry-breaking field). However, our method admits an extension to the case of the more general Hamiltonian

$$H_N(\{h\}) = \hat{H}_N - \sum_{i=1}^{N} h_i S_i \tag{2.39}$$

where $\hat{H}_N$ is given by (1.1) and h_i, $i = 1,..., N$, are independent identically distributed random variables. In this case our results are still valid, but now the symbol E denotes the expectation over the J_{ij} and h_i; the argument of $\ln 2\,\mathrm{ch}(\cdots)$ and $\mathrm{th}^2(\cdots)$ in (1.2) and (1.3) now is $Kq^{1/2}u + \beta h$ and the respective expressions contain also the integration over h with its distribution $\mu(dh)$. For example, Eq. (1.3) now has the form

$$q = \int \mathrm{th}^2(Kq^{1/2} + \beta h)\, e^{-\mu^2/2}(2\pi)^{-1/2}\, \mu(dh) \tag{2.40}$$

Of particular interest is the case of the Gaussian h_i with zero mean and the variance h_0^2. In this case Eq. (2.40) has a unique solution, which is not zero if $h_0 = 0$. Therefore the condition (2.11) of Theorem 1 can be omitted, because the only reason to impose this condition was to guarantee the convergence of q_N to a nonzero solution of Eq. (1.3), which for $\beta J > 1$ has two solutions: $q = 0$ and $q > 0$.

2. It also is not necessary to assume that the J_{ij} have zero mean value, because in that case also an expression for the limit free energy was "obtained" by Sherrington and Kirkpatrick[1] and we can generalize our previous results.

3. The normalization factor $N^{-1/2}$ in (1.1) allows us to consider other cases than Gaussian J_{ij}. For instance, we can extend our results to the case of i.i.d. J_{ij} that admit the estimate

$$E\{\exp(J_{ij}t)\} \leqslant \exp(Ct^2) \tag{2.41}$$

for some $C>0$ and $t<t_0$, $t_0>0$. An example of J_{ij} satisfying (2.41) is given by any symmetrically distributed J_{ij} whose distribution has a compact support (e.g., $J_{ij}=1$) or, more generally, J_{ij} for which $E\{J_{ij}^2 \exp(tJ_{ij})\}<\infty$, $|t| \leqslant t_0$. Proofs for these and more general distributions are more tedious, require additional arguments, and will be published elsewhere. We restrict ourselves here to the simplest, but fairly important case of Gaussian J_{ij} to make our results more transparent, because relations like (2.19), (3.13), etc., which are special to the Gaussian distribution, make the proofs substantially shorter.

3. AUXILIARY FACTS

Lemma 3.1. Let $\Delta_N(\beta)$ be defined by (2.10) and

$$\Delta_N(t, \beta) = E_G\{[\delta_N(t)\}^2\}$$

with $\delta_N(t)$ defined by (2.23) and (2.5). Then

$$\Delta_N(0, \beta) = \Delta_{N-1}(\beta_N) + O(N^{-1/4}), \qquad \text{where} \quad \beta_N = (1-N^{-1})^{1/2}\, \beta$$

Proof. Set

$$R_N = \gamma_N \sum_{i=2}^{N} v_i S_i - N^{-1/4} v_1 S_1, \qquad H_N^{(u)} = \alpha_N^2 H_{N-1} + uR_N$$

Then, in view of (2.15),

$$\begin{aligned} E_G\{\langle S_i\rangle^2_{H_N(0)}\} = E_G\{\langle S_i\rangle_{\alpha_N^2 H_{N-1}}\} &+ \int_0^1 du\, E_G\left\{2\langle S_i\rangle_u \left[\gamma_N \sum_2^N (\langle S_i S_k\rangle - \langle S_i\rangle_u \langle S_k\rangle_u)\, v_k\right.\right. \\ &\left.\left. - N^{-1/4}(\langle S_i S_1\rangle_u - \langle S_i\rangle_u \langle S_1\rangle_u)\right]\right\} \end{aligned}$$

where the symbol $\langle\cdots\rangle_u$ denotes the Gibbs average with the Hamiltonian $H_N^{(u)}$. The integrand in the last expression is $O(N^{-1/4})$ [see (2.3) and (2.16)]. Thus,

$$E_G\{|\langle S_i\rangle^2_{H_N(0)} - \langle S_i\rangle^2_{\alpha_N H_{N-1}}|\} = O(N^{-1/4})$$

Now we need only to take into account that the Gibbs average with the Hamiltonian $\alpha_N H_{N-1}$ at the temperature β coincides with the Gibbs average with H_{N-1} at the temperature $\alpha_N \beta$, and the lemma is proved.

Consider now the following Hamiltonian for the spins $S_2 \cdots S_N$:

$$H_N^{\pm}(t) = H_N(t)|_{S_1 = \pm 1} \tag{3.1}$$

where the $H_N(t)$ is given as in (2.14), and denote by $\langle \cdots \rangle_{\pm t}$ the respective Gibbs averages.

Lemma 3.2. Let

$$\delta_N^{\pm}(t) = N^{-1} \sum^{N} \langle S_k \rangle_{\pm t}^2 - q_{N-1}(\beta_N) \tag{3.2}$$

and

$$\Delta_N^{\pm}(t) = E_G\{[\delta^{\pm}(t)]^2\} \tag{3.3}$$

where $q_N(\beta)$ and β_N are defined in (2.10) and (2.17). Then under condition (2.12)

$$\lim_{N \to \infty} \Delta_N^{\pm}(t) = 0 \tag{3.4}$$

uniformly in $t \in [0, 1]$.

Proof. Since

$$\Delta_N^{\pm}(t) = \Delta_N^{\pm}(0) + \int_0^t \frac{d\Delta_N^{\pm}}{dt}(t)\, dt \tag{3.5}$$

and according to Lemma 3.1, (3.1), and (2.15), $\Delta_N^{\pm} \to 0$ as $N \to \infty$, it suffices to show that the second term in the rhs of (3.5) disappears as $N \to \infty$.

Denote temporarily $[\delta_N^{\pm}(t)]^2$ by a_N and take into account that in view of (3.1) and (2.14) this quantity depends on t, J_{1k}, and v_k for $k = 2,..., N$ via the combinations $\pm N^{-1/2} t J_{1k} + N^{-1/4} v_k$. Then, by using (2.1) and (2.19), we have

$$\begin{aligned}
\left|\frac{d\Delta_N^{\pm}}{dt}\right| &= \left| t^{-1} E_G \left\{ \sum_2^N J_{1k} \frac{da_N}{dJ_{1k}} \right\} \right| \\
&= \left| t J^2 E_G \left\{ \sum_2^N \frac{d^2 a_N}{dJ_{1k}^2} \right\} \right| \\
&= \left| t J^2 N^{-1/2} E_G \left\{ \sum_2^N \frac{d^2 a_N}{dv_k^2} \right\} \right| \\
&= |2tJ^2 N^{-1/2} E_G\{[G(v) - \delta(v)]\, a_N\}| \leqslant 2tJ^2 N^{-1/2}
\end{aligned}$$

Hence, the second term in (3.5) also disappears in the thermodynamic limit $N \to \infty$. This proves the lemma.

Lemma 3.3. Let $p_{\pm} = Z_N^{\pm}(t)/Z_N(t)$, where $Z_N^{\pm}(t)$ and $Z_N(t)$ are the partition functions of the Hamiltonians (3.1) and (2.14) and

$$\sigma_N(t) = N^{-1} \sum^{N} \langle S_k \rangle_{+t} \langle S_k \rangle_{-t} \tag{3.6}$$

Then

$$\lim_{N \to \infty} E_G \{ [\sigma_N(t) - q_{N-1}(\beta_N)]^2 \, p_+ p_- \} = 0$$

uniformly in $t \in [0, 1]$.

Proof.

$$\begin{aligned} E_G\{(\sigma_N - q_{N-1})^2 \, p_+ p_-\} &= E_G\{(\sigma_N^2 - q_{N-1}^2) \, p_+ p_-\} \\ &\quad - 2q_{N-1} E_G\{(\sigma_N - q_{N-1}) \, p_+ p_-\} \end{aligned} \tag{3.7}$$

But according to (3.6) and the Cauchy inequality, $\sigma_N^2 \leqslant \bar{q}_N^+ \bar{q}_N^-$, where $\bar{q}^{\pm}(t) = N^{-1} \sum_2^N \langle S_i \rangle_{\pm t}^2$. Thus,

$$\sigma_N^2 - q_{N-1}^2 \leqslant (\bar{q}_N^+ - q_{N-1}) \, q_{N-1} + (\bar{q}_N^- - q_{N-1}) \, q_N^+$$

and, in view of Lemma 3.1 and the inequalities $0 \leqslant q_{N-1}$ and $q_N^{\pm} \leqslant 1$, we conclude that

$$\lim_{N \to \infty} E_G\{(\sigma_N^2 - q_{N-1}^2)\} = 0$$

Let us show that the second term in (3.7) also tends to zero as $N \to \infty$. To this end, let us consider the expression

$$\Gamma_N = E_G \left\{ \sum_1^N \left(\left.\frac{d^2 F_N}{dv_i^2}\right|_t - \left.\frac{d^2 F_N}{dv_i^2}\right|_0 \right) \right\} \tag{3.8}$$

where $F_N(t)$ is defined by (2.33). Repeating almost literally the proof of Lemma 3.2, we obtain [cf. (2.34)] that

$$\begin{aligned} \Gamma_N &= \int_0^t d\tau \, E_G \left\{ \sum_1^N \frac{d^2}{dv_i^2} \left.\frac{dF_N}{d\tau}\right|_t \right\} \\ &= \beta N^{-1/2} \int_0^t d\tau \, E_G \left\{ \sum_1^N \frac{d^2}{dv_i^2} \sum_2^N J_{1i} \langle S_1 S_i \rangle_{H_N(\tau)} \right\} \\ &= -K^2 N^{-1} \int_0^t \tau \, d\tau \, E_G \left\{ \sum_1^N \frac{d^2}{dv_i^2} \sum_2^N \langle S_i S_1 \rangle_{H_N(\tau)}^2 \right\} \\ &= \int_0^t \tau \, d\tau \, E_G \left\{ [\delta(\tau) - G(v)] \sum_2^N \langle S_i S_1 \rangle_{H_N(\tau)}^2 \right\} \end{aligned}$$

Thus, $|\Gamma_N| \leqslant K^2$. On the other hand,

$$\frac{d^2F_N}{dv_i^2} = \beta^2 N^{-1/2}(1 - \langle S_i \rangle_{H_N(\tau)}^2)$$

and

$$\begin{aligned}\Gamma_N &= \beta^2 N^{-1/2} E_G \left\{ \sum_1^N \langle S_i \rangle_{H_N(0)}^2 - \langle S_i \rangle_{H_N(t)}^2 \right\} \\ &= \beta^2 N^{1/2} E_G \{ \delta_N(0) - p_+^2 \delta_N^+(t) - p_-^2 \delta_N^-(t) \\ &\quad - 2p_+ p_- [\sigma_N(t) - q_{N-1}(\beta_N)] \}\end{aligned}$$

where we took into account that according to (2.14) and (3.1),

$$p_+(t) + p_-(t) = 1, \qquad \langle S_i \rangle_{H_N(t)} = p_+(t) \langle S_i \rangle_{+t} + p_-(t) \langle S_i \rangle_{-t}$$

and

$$\langle \cdots \rangle_{\pm t}|_{t=0} = \langle \cdots \rangle_{H_N(0)}$$

Comparing now the two expressions obtained for Γ_N, we find the estimate

$$\begin{aligned}2\,|E_G\{\sigma_N(t) - q_{N-1}(\beta_N)\}| &\leqslant O(N^{-1/2}) + E_G\{|\delta_N(0)|\} \\ &\quad + E_G\{|\delta_N^+(t)|\} + E_G\{|\delta_N^-(t)|\}\end{aligned}$$

which in view of Lemmas 3.1 and 3.2 implies that the second term on the rhs of (3.7) also disappears in the thermodynamic limit. The proof of Lemma 3.3 is completed.

Proposition. Under condition (2.12) the second moments of the quantities (2.23) tend to zero as $N \to \infty$.

Proof. By using (3.8) and the similar relations

$$\langle S_k \rangle_{H_N(t)} = p_+ - p_- \langle S_1 S_k \rangle_{H_N(t)} = p_+ \langle S_k \rangle_{+t} - p_- \langle S_k \rangle_{-t} \tag{3.9}$$

we obtain for the quantity (2.23a)

$$\delta_N = p_+^2 \delta_N^+ + p_-^2 \delta_N^- + 2p_+ p_- (\sigma_N - q_{N-1}) \tag{3.10}$$

where $\delta_N^\pm(t)$ and $\sigma_N(t)$ are defined by (3.2) and (3.6). Thus

$$E_G\{\delta_N^2\} \leqslant 3E_G\{(\delta_N^+)^2\} + 3E_G\{(\delta_N^-)^2\} + 12E_G\{(\sigma_N - q_{N-1})^2\}$$

and according to Lemmas 3.2 and 3.3, all terms on the in rhs of this inequality tend to zero as $N \to \infty$. We have proved (2.23a). To prove (2.23b) and (2.23c), we apply similar arguments to the identities

$$\varepsilon_N = p_+^2 \delta_N^+ - p_-^2 \delta_N^-, \qquad \kappa = p_+^2 \delta_N^+ + p_-^2 \delta_N^+ - 2p_+ p_- \sigma_N$$

which follow easily from (2.23b), (2.23c). This proves the proposition.

Lemma 3.4. The operator A, defined in the Hilbert space of semi-infinite sequences with the norm (2.25) by operation (2.21), is nonnegative and self-ajoint.

Proof. Denote by A_0 the operator defined on the linear manifold D of sequences with finitely many nonzero elements by operation (2.21). It is easy to check that A_0 can be represented in the form

$$A_0 = D_1 R^* D_2 R \tag{3.11}$$

where $(D_1 x)_n = 2n x_n$, $D_2 = D_1 + 1$, $(Rx)_n = x_n - x_{n+1}$, and R^* is the operator ajoint to R. By using this representation, we find that

$$(A_0 x, x) = 2 \sum_{n=1}^{\infty} (2n+1)\,|x_n - x_{n+1}|^2 \geqslant 0$$

and that $(A_0 x, y) = (x, A_0 y)$. Thus, it suffices to establish that A_0 is essentially self-ajoint. By general principles,[12] the deficiency indices of A_0 may be $(0, 0)$ or $(1, 1)$. To exclude the latter possibility, we should show that there is no solution of the equation $\hat{\alpha} x = \lambda x$, $x_0 = 0$, which has the finite norm (2.25) when λ belongs to the resolvent set of A_0. Since A_0 is nonnegative, it suffices to prove this fact for a negative λ. Consider the solution $P_n(\lambda)$ which satisfies the conditions $P_0(\lambda) = 0$, $P_1(\lambda) = 1$. Then any solution $\{x_n\}_1^\infty$, $x_0 = 0$, of our equation has the form $x_n = x_1 P_n(\lambda)$. But in view of (3.11) we obtain that for $n \geqslant 1$

$$P_n(\lambda) = 1 + \sum_1^\infty (2k+1)^{-1} \sum_{l=1}^{k} (-\lambda)/(2l)\, P_l(\lambda)$$

Considering this formula successively for $n = 2, 3$, we find that $P_n(\lambda) \geqslant 1$ for $\lambda < 0$. Thus $\|P(\lambda)\| = \infty$ for $\lambda < 0$, and the lemma is proved.

Lemma 3.5. Let

$$\eta^{(N)} = \{n^2 (E_G \{m_1^{4(n-1)}(t)\})^{1/2}\}_{n=1}^\infty$$

Then

$$\|\eta^{(N)}\| \leqslant C < \infty$$

where the norm $\|\cdots\|$ is defined in (2.25) and C is independent of $t \in [0, 1]$ and N.

Proof. According to (2.24) and (2.25), we should prove that

$$E_G\left\{\sum_{n=1}^{\infty} n^3 m_1^{4n}(t)\right\} < \infty$$

uniformly in t and N. But since

$$\sum_{n=1}^{\infty} n^3 m_1^{4n} \leqslant (1 - m_1^4)^{-3}$$

and according to (2.5), (2.14), and (3.1),

$$\begin{aligned}1 - m_1^4 \geqslant 1 - m_1 &= 1 - [Z_N^+(t) - Z_N^-(t)][Z_N^+(t) + Z_N^-(t)]^{-1} \\ &= 2[1 + Z_N^-(t)/Z_N^+(t)]^{-1}\end{aligned}$$

it suffices to show that

$$E_G\{[Z_N^-(t)/Z_N^+(t)]^3\} < \infty$$

uniformly in t, N. By using (3.1), we have for

$$\varepsilon \equiv \exp\left(t\beta N^{-1/2} \sum_{k=2}^{N} J_{1k} S_k\right)$$

that

$$E_G\{(Z_N^-/Z_N^+)^3\} = E_G\{\langle\varepsilon\rangle_{H_N(0)}^3 \langle\varepsilon^{-1}\rangle_{H_N(0)}^3\} \leqslant E_G\{\langle\varepsilon\rangle_{H_N(0)}^6\}$$

But since $H_N(0)$ contains only the variables J_{ij}, $i, j \leqslant N-1$, which are statistically independent with J_{1k}, $k = 2,\ldots, N$, the rhs of the last inequality will be finite if

$$E_0\left\{\exp\left[\beta t N^{-1/2} \sum_{k=2}^{N} J_{1k} \sum_{\alpha=1}^{6} S_k^{(\alpha)}\right]\right\} < \infty \tag{3.12}$$

uniformly in t, N, and $S_k^{(\alpha)}$, where $S_k^{(\alpha)}$, $\alpha = 1,\ldots, 6$, are the "replicas" of the spins S_k, and $E_0\{\cdots\}$ denotes the expectation with respect to the family $\{J_{12},\ldots, J_{1N}\}$. By the independence of the J_{1k}, the lhs of (3.12) is

$$\prod_{k=2}^{N} E\left\{\exp\left[\beta t N^{-1/2} J_{1k} \sum_{\alpha=1}^{6} S_k^{(\alpha)}\right]\right\} \tag{3.13}$$

For the Gaussian J_{1k} factors in (3.13) are

$$\exp\left[\beta^2 t^2 J^2 (2N)^{-1} \left(\sum_{\alpha=1}^{6} S_k^{(\alpha)}\right)^2\right] \leqslant \exp(18\beta^2 J^2) \tag{3.14}$$

Combining (3.13) and (3.14), we obtain (3.12). This proves the lemma.

APPENDIX

Theorem 2. Let $f_N = (\beta N)^{-1} \ln Z_N$, where Z_N is the partition function of the SK Hamiltonian (1.1). Then

$$E\{(f_N - E\{f_N\})^2\} \leqslant C\beta^2 J^4 N^{-1} \tag{A.1}$$

where C is an absolute constant and $J^2 = E\{J_{ij}^2\}$.

Proof. Denote by J_k the family of random i.i.d. variables $J_{k1} \cdots J_{kk-1}$. by $\mathscr{F}_k$ the σ-algebra generated by J_k, $J_{k+1}, \ldots$, and by $F_N^{(k)} = E\{-\beta^{-1} \ln Z_N \mid \mathscr{F}_k\}$. Now, $F_N^{(k)}$ depends only on $J_k, \ldots, J_N$, and hence

$$F_N^{(1)} = -\beta^{-1} \ln Z_N, \qquad F_N^{(N+1)} = E\{-\beta^{-1} \ln Z_N\}$$
$$E\{F_N^{(k)} \mid \mathscr{F}_l\} = F_N^{(m)}, \qquad m = \max(k, l) \tag{A.2}$$

Therefore, if $\psi_k = F_N^{(k)} - F_N^{(k+1)}$, then

$$f_N - E\{f_N\} = N^{-1} \sum_{k=1}^{N} \psi_k$$
$$E\{(f_N - E\{f_N\})^2\} = N^{-2} \sum_{k=1}^{N} E\{\psi_k^2\} + 2N^{-2} \sum_{k<l} E\{\psi_k \psi_l\} \tag{A.3}$$

But according to (A.2), $E\{\psi_k \mid \mathscr{F}_l\} = 0$, $1 > k$, and since ψ_l is measurable with respect to $\mathscr{F}_l$, we have for $k < l$

$$E\{\psi_k \psi_l\} = E\{E\{\psi_k \psi_l \mid \mathscr{F}_l\}\} = E\{\psi_l E\{\psi_k \mid \mathscr{F}_l\}\} = 0 \tag{A.4}$$

Thus the second sum on the rhs of (A.3) is equal to zero and we have only to show that

$$E\{\psi_k^2\} \leqslant C < \infty \tag{A.5}$$

for some quantity C independent of k and N. To this end, consider the Hamiltonian $H_N^{(k)}(t)$ that results from (1.1) if one replaces the family

$J_k = \{J_{ik}\}_{i=1}^{k-1}$ by tJ_{ik}, $t \in [0, 1]$. Then $H_N^{(k)}(1) = H_N$, $H_N^{(k)}(0) = H_N|_{J_k=0}$, and

$$F_N = F_N|_{J_k=0} - N^{-1/2} \int_0^1 \left(\sum_{i<k} J_{ik} \langle S_i S_k \rangle_t \right) dt$$

where $\langle \cdots \rangle_t$ denotes the Gibbs average with the Hamiltonian $H_N^{(k)}(t)$. Let $E_k\{\cdots\}$ be the expectation with respect to the family J_k. Then, according to (A.2), $F_N^{(k+1)} = E_k\{F_N^{(k)}\}$ and

$$E\{\psi_k^2\} = E\{E_k\{\psi_k^2\}\} = E\{E_k\{B_N^2\} - E_k^2\{B_N\}\} \tag{A.6}$$

where

$$B_N = -\int_0^1 \left(\sum_{i<k} J_{ik} \langle S_i S_k \rangle_t \right) dt$$

By using the identity (2.19), we find that

$$E_k\{B_N^2\} = \beta J^2 N^{-1} \int_0^1 \left[\sum_{i<k} (1 - \langle S_i S_k \rangle_t^2 \right] t\, dt \tag{A.7}$$

The same identity and simple, but somewhat tedious calculations give

$$E_k\{B_N^2\} = E^2\{B_N\} + \beta^2 K^2 N^{-2} \int^1 C_{N,k}(t_1, t_2)\, t_1 t_2\, dt_1\, dt_2 \tag{A.8}$$

where $C_{N,k}(t_1, t_2)$ is the double sum over $1 \leqslant i, j < k$ of the products of the Gibbs averages with $H_N^{(k)}(t_1)$ and $H_N^{(k)}(t_2)$. There are nine such products, e.g., $\langle S_i S_k \rangle_1 \langle S_j S_k \rangle_2 \langle S_i S_l \rangle_1 \langle S_i S_j \rangle_2$, $\langle S_i S_k \rangle_1 \langle S_j S_k \rangle_2$, etc. $[\langle \cdots \rangle_{1,2}$ denotes the Gibbs average with $H_N^{(k)}(t_{1,2})]$. Each of them is obviously bounded by an absolute constant. Therefore the second term on the rhs of (A.8) is bounded above by the expression $C\beta^2 J^4 N^{-1}$, where C is an absolute constant. Combining this fact with (A.6), we obtain (A.5), which with (A.4) and (A.3) yields the assertion of Theorem 2. By developing these arguments, one can also prove the central limit theorem for f_N.

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